

Study of a Fractional Function Equation

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Abstract: In this paper, based on a new multiplication of fractional analytic functions, we study a fractional function equation. Using some methods, we can obtain the solution of this fractional function equation. In fact, the solution is a generalization of the result in traditional function equation.

Keyword: New multiplication, Fractional analytic functions, Fractional function equation.

I. INTRODUCTION

In 1695, the concept of fractional calculus first appeared in a famous letter between L'Hospital and Leibniz. Many great mathematicians have further developed this field. We can mention Euler, Lagrange, Laplace, Fourier, Abel, Liouville, Riemann, Hardy, Littlewood, and Weyl. In the past decades, fractional calculus has been considered as one of the best tools to describe the process of long memory. Such models are interesting for physicists, engineers, and mathematicians. Fractional calculus has important applications in different fields such as physics, mechanics, dynamics, electrical engineering, viscoelasticity, biology, economics, control theory, and so on [1-10].

In this paper, based on a new multiplication of fractional analytic functions, we find the solution of the following α -fractional function equation:

$$f_\alpha(x^\alpha) + f_\alpha\left(\left[1 - \frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes -1}\right) = \frac{1}{\Gamma(\alpha+1)}x^\alpha, \tag{1}$$

where $0 < \alpha \leq 1$, $\Gamma(\cdot)$ is the gamma function, and $\frac{1}{\Gamma(\alpha+1)}x^\alpha \neq 0$, $\frac{1}{\Gamma(\alpha+1)}x^\alpha \neq 1$. In fact, our result is a generalization of classical function equation result.

II. PRELIMINARIES

Definition 2.1 ([11]): Suppose that x and a_k are real numbers for all k , and $0 < \alpha \leq 1$. If the function $f_\alpha: [a, b] \rightarrow R$ can be expressed as an α -fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}x^{k\alpha}$, then we say that $f_\alpha(x^\alpha)$ is α -fractional analytic function.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.2 ([12]): If $0 < \alpha \leq 1$, and x_0 is a real number. Let $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ be two α -fractional analytic functions,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}x^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}, \tag{2}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}x^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}. \tag{3}$$

Then

$$\begin{aligned}
 & f_{\alpha}(x^{\alpha}) \otimes g_{\alpha}(x^{\alpha}) \\
 &= \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} x^{k\alpha} \otimes \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} x^{k\alpha} \\
 &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \left(\sum_{m=0}^k \binom{k}{m} a_{k-m} b_m \right) x^{k\alpha}.
 \end{aligned} \tag{4}$$

Equivalently,

$$\begin{aligned}
 & f_{\alpha}(x^{\alpha}) \otimes g_{\alpha}(x^{\alpha}) \\
 &= \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right)^{\otimes k} \otimes \sum_{k=0}^{\infty} \frac{b_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right)^{\otimes k} \\
 &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{m=0}^k \binom{k}{m} a_{k-m} b_m \right) \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right)^{\otimes k}.
 \end{aligned} \tag{5}$$

Definition 2.3 ([13]): If $0 < \alpha \leq 1$, and $f_{\alpha}(x^{\alpha})$, $g_{\alpha}(x^{\alpha})$ be two α -fractional analytic functions defined on an interval containing x_0 ,

$$f_{\alpha}(x^{\alpha}) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} x^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right)^{\otimes k}, \tag{6}$$

$$g_{\alpha}(x^{\alpha}) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} x^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right)^{\otimes k}. \tag{7}$$

We define the compositions of $f_{\alpha}(x^{\alpha})$ and $g_{\alpha}(x^{\alpha})$ by

$$(f_{\alpha} \circ g_{\alpha})(x^{\alpha}) = f_{\alpha}(g_{\alpha}(x^{\alpha})) = \sum_{k=0}^{\infty} \frac{a_k}{k!} (g_{\alpha}(x^{\alpha}))^{\otimes k}, \tag{8}$$

and

$$(g_{\alpha} \circ f_{\alpha})(x^{\alpha}) = g_{\alpha}(f_{\alpha}(x^{\alpha})) = \sum_{k=0}^{\infty} \frac{b_k}{k!} (f_{\alpha}(x^{\alpha}))^{\otimes k}. \tag{9}$$

Definition 2.4 ([14]): Suppose that $0 < \alpha \leq 1$, and $f_{\alpha}(x^{\alpha})$, $g_{\alpha}(x^{\alpha})$ are two α -fractional analytic functions. Then $(f_{\alpha}(x^{\alpha}))^{\otimes n} = f_{\alpha}(x^{\alpha}) \otimes \dots \otimes f_{\alpha}(x^{\alpha})$ is called the n th power of $f_{\alpha}(x^{\alpha})$. On the other hand, if $f_{\alpha}(x^{\alpha}) \otimes g_{\alpha}(x^{\alpha}) = 1$, then $g_{\alpha}(x^{\alpha})$ is called the \otimes reciprocal of $f_{\alpha}(x^{\alpha})$, and is denoted by $(f_{\alpha}(x^{\alpha}))^{\otimes -1}$.

III. MAIN RESULT

In this section, we will solve a fractional function equation.

Theorem 3.1: If $0 < \alpha \leq 1$, and $\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \neq 0$, $\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \neq 1$, then the solution of the α -fractional function equation

$$f_{\alpha}(x^{\alpha}) + f_{\alpha} \left(\left[1 - \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes -1} \right) = \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \tag{10}$$

is

$$f_{\alpha}(x^{\alpha}) = \left[\left(\left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes 3} - \frac{1}{\Gamma(\alpha+1)} x^{\alpha} + 1 \right) \otimes \left[2 \cdot \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \otimes \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} - 1 \right) \right]^{\otimes -1} \right]. \tag{11}$$

Proof Replace $\frac{1}{\Gamma(\alpha+1)} x^{\alpha}$ with $\left[1 - \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes -1}$ in Eq. (10), then

$$\begin{aligned}
 & \left[1 - \left[1 - \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes -1} \right]^{\otimes -1} \\
 &= \left[\left(1 - \frac{1}{\Gamma(\alpha+1)} x^{\alpha} - 1 \right) \otimes \left[1 - \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes -1} \right]^{\otimes -1} \\
 &= \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} - 1 \right) \otimes \left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes -1}.
 \end{aligned} \tag{12}$$

Therefore,

$$f_{\alpha} \left(\left[1 - \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes -1} \right) + f_{\alpha} \left(\left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} - 1 \right) \otimes \left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes -1} \right) = \left[1 - \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes -1}. \quad (13)$$

Replace $\frac{1}{\Gamma(\alpha+1)} x^{\alpha}$ with $\left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} - 1 \right) \otimes \left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes -1}$ in Eq. (10), then

$$\begin{aligned} & \left[1 - \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} - 1 \right) \otimes \left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes -1} \right]^{\otimes -1} \\ &= \left[\left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} - \frac{1}{\Gamma(\alpha+1)} x^{\alpha} + 1 \right) \otimes \left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes -1} \right]^{\otimes -1} \\ &= \frac{1}{\Gamma(\alpha+1)} x^{\alpha}. \end{aligned} \quad (14)$$

Hence,

$$f_{\alpha} \left(\left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} - 1 \right) \otimes \left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes -1} \right) + f_{\alpha}(x^{\alpha}) = \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} - 1 \right) \otimes \left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes -1}. \quad (15)$$

From Eq. (10) plus Eq. (15) minus Eq. (13), we get

$$\begin{aligned} & 2f_{\alpha}(x^{\alpha}) \\ &= \frac{1}{\Gamma(\alpha+1)} x^{\alpha} + \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} - 1 \right) \otimes \left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes -1} - \left[1 - \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes -1} \\ &= \left(\left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes 2} + \frac{1}{\Gamma(\alpha+1)} x^{\alpha} - 1 \right) \otimes \left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes -1} + \left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha} - 1 \right]^{\otimes -1} \\ &= \left[\left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} - 1 \right) \otimes \left(\left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes 2} + \frac{1}{\Gamma(\alpha+1)} x^{\alpha} - 1 \right) + \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right] \otimes \left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \otimes \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} - 1 \right) \right]^{\otimes -1} \\ &= \left[\left(\left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes 3} - \frac{1}{\Gamma(\alpha+1)} x^{\alpha} + 1 \right) \right] \otimes \left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \otimes \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} - 1 \right) \right]^{\otimes -1}. \end{aligned} \quad (16)$$

Finally, we have

$$f_{\alpha}(x^{\alpha}) = \left[\left(\left[\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right]^{\otimes 3} - \frac{1}{\Gamma(\alpha+1)} x^{\alpha} + 1 \right) \right] \otimes \left[2 \cdot \frac{1}{\Gamma(\alpha+1)} x^{\alpha} \otimes \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} - 1 \right) \right]^{\otimes -1}.$$

Q.e.d.

IV. CONCLUSION

Based on a new multiplication of fractional analytic functions, this paper studies a fractional function equation. In fact, the fractional function equation is a generalization of ordinary function equation. Using some methods, we obtain the solution of this fractional function equation. In the future, we will continue to use the new multiplication of fractional analytic functions to solve the problems in fractional calculus and applied mathematics.

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